Math 155, ***Lecture Notes- Bonds*** **Name**____________

Section 9.6 The Ratio and Root Tests

The Ratio and Root test will test for absolute convergence. The Ratio Test is useful for studying series that have factorials, or exponentials. The Root Test is useful for studying series that have *nth* powers.

THEOREM 9.17 Ratio Test

Let Σa_n be a series with nonzero terms.

1. Σa_n converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.
2. Σa_n diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Ex. 1: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4}{(2n)!}$

Let $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4}{(2n)!} = \sum_{n=1}^{\infty} a_n$, with $a_n = \frac{(-1)^{n+1}4}{(2n)!}$ and $a_{n+1} = \frac{(-1)^{(n+1)+1}4}{[2(n+1)]!}$

$a_{n+1} = \frac{(-1)^{n+2}4}{(2n+2)!}$.

We can see $a_n \neq 0$ for all $n \geq 1$, since $\frac{(-1)^{n+1}4}{(2n)!} \neq 0$.

Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}4}{(2n+2)!}}{\frac{(-1)^{n+1}4}{(2n)!}}\right|$

$= \lim_{n\to\infty}\frac{4}{(2n+2)!}\cdot\frac{(2n)!}{4}$

$= \lim_{n\to\infty}\frac{(2n)!}{(2n+2)\cdot(2n+1)\cdot[(2n)!]}$

$= \lim$

More Ex. 1:

$$= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)}$$

$= 0$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the Ratio Test tells us that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{(2n)!}$ converges absolutely. $\square$

Ex. 2: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Let $\sum_{n=1}^{\infty} \frac{n^n}{n!} = \sum_{n=1}^{\infty} a_n$, with $a_n = \frac{n^n}{n!}$ and $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$.

We can see $a_n \neq 0$ for all $n \geq 1$, since $\frac{n^n}{n!} \neq 0$.

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right|$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)} \cdot (n+1)^n \cdot \cancel{n!}}{\cancel{(n+1)} \cdot \cancel{(n!)} \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

$= e$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = e > 1$, the Ratio Test tells us that $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges. $\square$

Ex. 3: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$

Let $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} a_n$, with $a_n = n\left(\frac{3}{2}\right)^n$ and $a_{n+1} = (n+1)\left(\frac{3}{2}\right)^{n+1}$.

We can see that $a_n \neq 0$ for all $n \geq 1$, since $n\left(\frac{3}{2}\right)^n \neq 0$.

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\left(\frac{3}{2}\right)^{n+1}}{n\left(\frac{3}{2}\right)^n}\right|$

$= \lim_{n\to\infty} \frac{(n+1)\cdot\left(\frac{3}{2}\right)}{n}$

$= \frac{3}{2}\cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$

$= \frac{3}{2}\cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$

$= \frac{3}{2}\cdot(1+0)$

$= \frac{3}{2}$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{2} > 1$, the Ratio Test tells us that $\sum_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$ diverges. $\square$

Ex. 4: Determine the convergence or divergence of the series: $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$

Let $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n} = \sum_{n=0}^{\infty} a_n$, with $a_n = \frac{3^n}{(n+1)^n}$ and $a_{n+1} = \frac{3^{n+1}}{[(n+1)+1]^{n+1}}$

$$a_{n+1} = \frac{3^{n+1}}{(n+2)^{n+1}}.$$

We can see that $a_n \neq 0$ for all $n \geq 1$, since $\frac{3^n}{(n+1)^n} \neq 0$.

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{(n+2)^{n+1}}}{\frac{3^n}{(n+1)^n}} \right|$

$$= \lim_{n\to\infty} \frac{3^{n+1} \cdot (n+1)^n}{(n+2)^{n+1} \cdot 3^n}$$

$$= \lim_{n\to\infty} \frac{3 \cdot (n+1)^n}{(n+2) \cdot (n+2)^n}$$

$$= 3 \cdot \lim_{n\to\infty} \left(\frac{1}{n+2}\right) \cdot \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n \quad \underline{\star\star\star}$$

$$= 3 \cdot 0 \cdot \left(\frac{1}{e}\right)$$

$= 0$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the Ratio Test tells us that $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$ converges absolutely. $\square$

$\underline{\star\star\star}$ - $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = \lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x$

let $y = \lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x$

$$\ln(y) = \ln\left[\lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x\right]$$

$$\ln(y) = \lim_{x\to\infty} \left[x \cdot \ln\left(\frac{x+1}{x+2}\right)\right]$$

$$\ln(y) = \lim_{x\to\infty} \left[\frac{x \cdot \ln\left(\frac{x+1}{x+2}\right)}{1}\right] \cdot \left[\frac{\frac{1}{x}}{\frac{1}{x}}\right]$$

More Ex. 3:

$$\ln(y) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x+2}\right)}{\frac{1}{x}} \rightarrow \frac{\ln\left(\frac{\infty}{\infty}\right)}{\frac{1}{\infty}} = \frac{\ln(1)}{0} = \boxed{\frac{0}{0}}$$

Indeterminate Form stop!

use L'Hôpital's Rule

$$\ln(y) = \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\ln\left(\frac{x+1}{x+2}\right)\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]}$$

$$\ln(y) = \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\ln(x+1) - \ln(x+2)\right]}{-\frac{1}{x^2}}$$

$$\ln(y) = \lim_{x\to\infty} \frac{\frac{1}{x+1} - \frac{1}{x+2}}{\frac{-1}{x^2}}$$

$$\ln(y) = \lim_{x\to\infty} \left[\frac{(x+2)-(x+1)}{(x+1)(x+2)}\right] \cdot \left[\frac{-x^2}{1}\right]$$

$$\ln(y) = \lim_{x\to\infty} \frac{-x^2}{x^2+3x+2}$$

$$\ln(y) = \lim_{x\to\infty} \left[\frac{\frac{-x^2}{1}}{x^2+3x+2}\right] \cdot \left[\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right]$$

$$\ln(y) = \lim_{x\to\infty} \frac{-1}{1+\frac{3}{x}+\frac{2}{x^2}}$$

$$\ln(y) = -1$$

$$e^{\ln(y)} = e^{-1}$$

$$\boxed{y = \frac{1}{e}} \quad \& \quad \boxed{\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n = \frac{1}{e}}$$

THEOREM 9.18 Root Test

Let Σa_n be a series.

1. Σa_n converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

2. Σa_n diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.

3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Ex. 5: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty}\left[\frac{\ln(n)}{n}\right]^{n}$

Let $\sum_{n=1}^{\infty}\left[\frac{\ln(n)}{n}\right]^{n} = \sum_{n=1}^{\infty} a_n$, with $a_n = \left[\frac{\ln(n)}{n}\right]^n$.

Consider $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left[\frac{\ln(n)}{n}\right]^n\right|}$

$$= \lim_{n\to\infty}\frac{\ln(n)}{n}$$

Indeterminate Form

Stop!

$$= \lim_{x\to\infty}\frac{\ln(x)}{x} \longrightarrow \frac{\ln(\infty)}{\infty} = \boxed{\frac{\infty}{\infty}}$$

Use L'Hôpital's Rule

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x]}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1}$$

$= 0$. Since $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 0 < 1$, the Root Test tells us that $\sum_{n=1}^{\infty}\left[\frac{\ln(n)}{n}\right]^n$ converges absolutely. $\square$

Ex. 6: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$

Let $\sum_{n=1}^{\infty}\left[\frac{-3n}{2n+1}\right]^{3n} = \sum_{n=1}^{\infty} a_n$, with $a_n = \left[\frac{-3n}{2n+1}\right]^{3n}$.

Consider $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left[\frac{-3n}{2n+1}\right]^{3n}\right|}$

$$= \lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)^{3}$$

$$= \left[\lim_{n\to\infty}\frac{3n}{2n+1}\right]^{3}$$

$$= \left(\lim_{n\to\infty}\left[\frac{\frac{3n}{1}}{\frac{2n+1}{1}}\right]\cdot\left[\frac{\frac{1}{n}}{\frac{1}{n}}\right]\right)^{3}$$

$$= \left(\lim_{n\to\infty}\frac{3}{2+\frac{1}{n}}\right)^{3}$$

$$= \left(\frac{3}{2+0}\right)^{3}$$

$= \frac{27}{8}$. Since $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \frac{27}{8} > 1$, the Root Test tells us that $\sum_{n=1}^{\infty}\left(\frac{-3n}{2n+1}\right)^{3n}$ diverges. $\square$

Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)\,dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

Ex. 7: Determine the convergence or divergence of the following series, and state the most efficient test that would show your result:

(a) $\sum_{n=1}^{\infty} \frac{n+1}{3n+1} \approx \sum_{n=1}^{\infty} \frac{n}{3n} \approx \sum_{n=1}^{\infty} \frac{1}{3}$, Use nth Term Test for Divergence, $\lim_{n\to\infty} a_n = \frac{1}{3} \neq 0$

(b) $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n = \sum_{k=0}^{\infty} \frac{\pi}{6}\cdot\left(\frac{\pi}{6}\right)^k = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. Geometric Series, converges $r=\frac{\pi}{6}$, $0<|r|<1$

(c) $\sum_{n=1}^{\infty} ne^{-n^2}$, Integral Test, let $f(x)=xe^{-x^2}$, consider $\int_1^{\infty} xe^{-x^2}dx$, converges

(d) $\sum_{n=1}^{\infty} \frac{1}{3n+1}$, General Harmonic Series, diverges

(e) $\sum_{n=1}^{\infty} (-1)^n\left(\frac{3}{4n+1}\right)$, Alternating Series Test, converges

(f) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$, Ratio Test, diverges

(g) $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$, Root Test, converges

Ex. 8: Determine the convergence or divergence of the series:

Consider $\sum_{n=1}^{\infty} a_n$, where $a_1 = \frac{1}{3}$ and $a_{n+1} = \left(1+\frac{1}{n}\right)a_n$.

Let $\sum_{n=1}^{\infty} a_n$ represent a series where $a_1 = \frac{1}{3}$ and $a_{n+1} = (1+\frac{1}{n})a_n$.

We can see that $a_n \neq 0$ for all $n \geq 1$, since $a_{n+1} = (1+\frac{1}{n})a_n$, with $a_1 = \frac{1}{3}$.

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(1+\frac{1}{n})a_n}{a_n}\right|$

$= \lim_{n\to\infty} (1+\frac{1}{n})$

$= 1+0$

$= 1.$ Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is <u>inconclusive</u>! Darn!

Consider

$$a_{n+1} = (1+\tfrac{1}{n})a_n$$
$$a_{n+1} = 1\cdot a_n + \tfrac{1}{n}\cdot a_n$$
$$a_{n+1} = a_n + \tfrac{1}{n}\cdot a_n.$$

This means that $a_{n+1} \geq a_n$ for all $n \geq 1$, and a_n is an increasing sequence. Since $a_1 = \frac{1}{3}$ and a_n is increasing, we know $\lim_{n\to\infty} a_n > 0$, which means that $\lim_{n\to\infty} a_n \neq 0$. According to the nth-Term Test for Divergence, the series $\sum_{n=1} a_n$ diverges.

$\square$